WAVE EVOLUTION IN A NONUNIFORM DISPERSE MEDIUM

V. Ts. Gurovich and R. K. Mazitov

Zhurnal Prikladnoi Makhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 6, pp. 153–154, 1968

The structure of nonlinear waves in a uniform disperse medium has been studied quite thoroughly [1–4]. The equations describing nonlinear waves have continuous periodic solutions corresponding to periodic steady-state waves. There also exists a physically meaningful solution in which the derivative is discontinuous at one point. The profile of this solution is symmetric about the discontinuity and corresponds to an isolated pulse (an isolated wave).

For the case of weak dispersion, the system of equations reduces to a single equation whose steady-state solution, which corresponds to an isolated wave, was found by Cortweg-de Brise, and whose transient self similar solution was found by Berezin and Karpman [4].

It was noted in [2] that an account of dissipation disrupts the symmetry of the isolated wave and causes the formation of a damped wave train. It is shown below that a similar situation occurs when there is no dissipation if the nonlinear wave is propagating in a nonuniform medium.

1. We consider the evolution of a magnetosonic wave excited in a plasma by a magnetic piston. We restrict the discussion to the case in which $H_0^2/8\pi \gg n_0 mc^2$, and we take into account the deviation of the plasma from quasi neutrality.

The system of equations describing this wave is analogous to that derived in [1, 3] for a uniform medium. The only difference is in the normalization of the magnetic field:

$$H - H_0(t) = (4\pi e n_0(0)/H_0(0))\,[\varphi - \varphi_0(t)]. \tag{1.1}$$

Here $H_0(t)$ and $\varphi_0(t)$ are the magnetic field intensity and the electric field potential ahead of the shock wave at an arbitrary time; $H_0(0)$ and $\varphi_0(0)$ are the same quantities at $t = 0$; $n_0(t)$ is the ion density at the wave front; and $n_0(0)$ is the ion density at $t = 0$.

The time dependences of H_0 and φ_0 are due to the nonuniformity of the medium. We set

$$H_0(t) = (4\pi e n_0(0)/H_0(0))\varphi_0(t). \tag{1.2}$$

We use the following dimensionless variables:

$$q = \frac{n}{n_0(0)}, \quad u = \left(\frac{8\pi n_0(0)M}{H_0^2(0)}\right)^{1/2} v, \quad \psi = \frac{4\pi e n_0(0)}{H_0^2(0)}\varphi,$$
$$\tau = \left(\frac{4\pi e^2 n_0(0)}{M}\right)^{1/2} t, \quad y = \left(\frac{4\pi e n_0(0)}{H_0(0)}\right)^{1/2} x, \tag{1.3}$$

where n is the ion density and M is the ion mass. Then the initial system of equations has the form

$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial y} = -\frac{\partial \psi}{\partial y}, \quad \frac{\partial q}{\partial \tau} + \frac{\partial (qu)}{\partial y} = 0, \quad \frac{\partial^2 \psi}{\partial y^2} = \psi - q. \tag{1.4}$$

At a sufficiently great distance from the piston in a nonuniform medium, the wave is almost independent of the initial conditions and may be assumed self-similar. On the basis of dimensionality considerations, we specify the law of motion of the wave front [5]:

$$\frac{dY}{d\tau} = -\frac{u_0}{1-\alpha\tau}, \qquad Y(\tau) = \frac{u_0}{\alpha}\ln|1-\alpha\tau|, \tag{1.5}$$

where α characterizes the medium nonuniformity, and u_0 is the wave velocity at $t = 0$.

In the coordinate system moving with the wave front, we seek the self-similar solution in the form

$$w = \frac{W(\eta)}{1-\alpha\tau}, \quad \psi = \frac{\Psi(\eta)}{(1-\alpha\tau)^2}, \quad q = \frac{Q(\eta)}{(1-\alpha\tau)^2}, \quad w = u + u_0. \tag{1.6}$$

Here $\eta = y - Y(\tau)$ is the self-similar variable. Significantly, the profiles of all the quantities behind the wave front remain constant in time; only their amplitudes change.

Substituting (1.6) into (1.4), we find a system of equations for the representatives W, Ψ, and Q:

$$\alpha(W - u_0) + \frac{d}{d\eta}\left(\frac{W^2}{2} + \Psi\right) = 0, \quad 2\alpha Q + \frac{d}{d\eta}(QW) = 0, \quad \frac{d^2\Psi}{d\eta^2} - \Psi = -Q. \tag{1.7}$$

The boundary conditions for W, Ψ, and Q are specified ahead of the wave front, at the point $\eta = 0$:

$$W(0) = 1, \ Q(0) = 1, \ \Psi(0) = 1, \ \Psi'(0) = 0 \tag{1.8}$$

It follows from (1.5), (1.6), and (1.8) that the wave propagates through the plasma with an exponentially increasing density $n \sim n_0(0)\exp(-2\alpha y/u_0)$ toward negative y.

2. We now discuss the case in which the nonuniformity is much larger than the dispersion length $\alpha \ll 1$. The presence of the small parameter α permits us to use a special perturbation theory for solving system (1.7) on the interval $0 \le \eta \le \alpha^{-1}$, which corresponds to the most interesting region near the wave front.

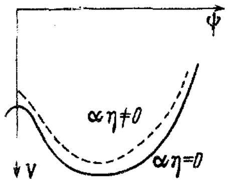

Fig. 1

Using the method of joined expansions worked out in [6], we find, with an accuracy of order α, the approximate integrals of the first two equations of systems (1.7) are

$$Q = u_0(W + 2\alpha\eta)^{-1}, \quad W = [u_0^2 + 2(1 - \Psi)]^{1/2} - \alpha\eta. \tag{2.1}$$

Substituting (2.1) into the third equation of (1.7), we find the following equation for the potential:

$$\frac{d^2\Psi}{d\eta^2} = \Psi - \frac{u_0}{[u_0^2 + 2(1-\Psi)]^{1/2} + \alpha\eta}. \tag{2.2}$$

Equation (2.2) is equivalent to the equation of motion of a nonlinear oscillator with a slowly varying potential energy:

$$V = -\tfrac{1}{2}\Psi^2 - u_0[u_0^2 + 2(1-\Psi)]^{1/2} + u_0\alpha\eta \ln|[u_0^2 + 2(1-\Psi)]^{1/2} + \alpha\eta|. \tag{2.3}$$

By analogy with the nonlinear oscillator, we can establish the wave structure.

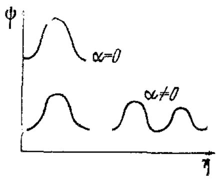

Fig. 2

In the case of a uniform medium $\alpha = 0$, oscillatory motion with an infinite period (and energy $E = -u_0^2 - 1/2$) corresponds to an isolated wave. When there is dissipation, the oscillator undergoes damped oscillations with a finite period. This means that a damped wave train arises behind the front [2, 4].

In the case of a nonuniform medium, the potential energy profile $V(\Psi)$ (Fig. 1) rises with increasing η; this is equivalent to a decrease of the oscillator energy.

The nonuniformity of the medium thus leads to a wave of oscillator structure, with the individual maxima separated by a distance of the order of the dispersion length $\lambda = H_0/4\pi n_0 e$ (Fig. 2).

The authors thank R. G. Sagdeev for useful advice.

REFERENCES

1. A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, "Nonlinear oscillations in a rarefied plasma," Yadernyi sintez, vol. 1, p. 82, 1961.
2. R. Z. Sagdeev, "Collective processes and shock waves in a plasma," collection: Problems of Plasma Theory [in Russian], no. 4, Atomizdat, Moscow, 1964.
3. C. E. Longmire, Elementary Plasma Physics [Russian translation], ch. 7, Atomizdat, Moscow, 1966.
4. Yu. A. Berezin and V. I. Karpman, "Theory of transient waves of finite amplitude in a rarefied plasma," ZhETF, vol. 46, p. 1880, 1964.
5. Ya. B. Zel'dovich and Yu. P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena [in Russian], ch. 12, Fizmatgiz, Moscow, 1963.
6. M. D. Van Dyke, Perturbation Methods in Fluid Mechanics [Russian translation], ch. 5, Mir, Moscow, 1967.

27 August 1968

Novosibirsk